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Ramp-induced wave-number selection for traveling waves

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The problem of wave-number selection by a ramp, i.e., a region smoothly matching sub- and supercritical domains, is considered within the framework of the cubic and quintic Ginzburg-Landau (GL) equations with a sign-changing overcriticality parameter. A local frequency is also allowed to be a smooth function of the spatial coordinate. For the cubic model, a unique value of the selected wave number is found by means of an asymptotic procedure valid when the imaginary parts of coefficients in the GL equation are small, while the group velocity is arbitrary. Under certain conditions, the wave number may lie outside the stability band, which is expected to give rise to a dynamical chaos. In the quintic model, which describes a system with the inverted bifurcation, the selection scenario is much simpler: A front separating a traveling wave and the trivial state is expected to be pinned at the point of the ramp where its velocity, regarded as a function of the local overcriticality, vanishes. Eventually, the wave-number selection is performed by the pinned front. Experimentally, the selection may be realized in the traveling-wave convection, or in a self-oscillatory chemical system.

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As is well known [1,2], the Eckhaus stability criterion singles out a whole band of wave numbers of stable patterns, which give rise to the famous wave-number selection problem. A solution to this problem was put forward in Ref. [3]: In a spatially inhomogeneous system with a so-called ramp of the overcriticality parameter ϵ, when $\epsilon(x)$ is negative at $x<0$ and positive at $x>0$, the system unambiguously selects the center of the Eckhaus stability band [4]. The objective of the present work is to consider a similar wave-number selection problem for waves produced by the oscillatory instability in a system with the ramped overcriticality. It should be feasible to realize this situation experimentally in the binary-fluid convection, which is well known to have the traveling-wave character [5]. To render the wave system effectively one dimensional, one can use a narrow channel, giving a small gradient along the channel to the temperature difference driving the convection. The important fact is that the traveling-wave convection experiments in the narrow channel can be conducted under very well-controlled conditions, which makes it possible to observe really subtle dynamical effects [6–8]. Another example of a well-controlled nonequilibrium system in which it seems possible to create the necessary gradient of the overcriticality is furnished by self-oscillatory chemical reactions in a gel host medium, where, in particular, a steady Turing-like pattern has been recently observed [9].

The analysis of the selection problem will be started in terms of the cubic Ginzburg-Landau (GL) equation, which is a commonly known phenomenological model of nonequilibrium systems with the oscillatory instability. However, the real traveling-wave convection is bistable, its simplest model being the quintic GL equation [8]. The selection problem for the latter model is briefly considered at the end of the present work. Although the quintic GL equation is, generally speaking, more complicated than the cubic one, a solution of this problem proves to be much simpler for the quintic equation. Anyway, the wave number is demonstrated to be always selected uniquely.

The cubic GL equation for the complex order parameter $u(x,t)$ is taken in the form

$$u_t+cu_x=\epsilon(x)u-i\chi(x)u-(1+i\alpha)|u|^2u+(1+i\beta)u_{xx}\ , \tag{1}$$

where c is the group velocity, α and β are coefficients of the nonlinear and spatial dispersion, and the local frequency χ (the imaginary part of the overcriticality ϵ) is allowed to be a smooth function of x together with the ramped $\epsilon(x)$. One can take, for instance,

$$\epsilon(x)=\tanh(x/l) \tag{2}$$

with $l\gg 1$. In what follows, it will be adopted, for convenience, that $\epsilon(+\infty)=1$ and $\chi(+\infty)=0$.

Traveling-wave solutions to Eq. (1) are looked for in the form

$$u(x,t)=a(x)\exp[i\phi(x)-i\omega t]\ . \tag{3}$$

The amplitude $a(x)$ and phase $\phi(x)$ are determined by the real equations ensuing from Eq. (1):

$$\epsilon(x)a-ca'-a^2+a''-ak^2-\beta ak'-2\beta a'k=0\ , \tag{4a}$$

$$-\omega a^2+ca^2k=a^2k'+2aa'k-\chi(x)a^2-\alpha a^4+\beta aa''-\beta a^2k^2\ , \tag{4b}$$

where $k(x)\equiv d\phi/dx$ is the local wave number, and the prime stands for d/dx. To attack the wave-number selection problem analytically, it is natural to develop a perturbative analysis, assuming the dispersion coefficients α and β small. Note that, for the real traveling-wave convection [6–8], β is always small, while α is not (typically, $\alpha\sim 5$). However, I have to adopt the assumption

$|\alpha| \ll 1$ to be able to solve the problem analytically. The results to be obtained below can give at least a qualitative understanding of what may happen at nonsmall α. A detailed analysis of the situation with $\alpha \geq 1$ should rely upon numerical simulations of Eq. (1), which are now underway [10].

If the dispersion parameters are assumed small, it is natural to assume as well that also the inhomogeneous frequency $\chi(x)$ is small. This assumption will be adopted below; however, it will be demonstrated then that $\chi(x) \neq 0$ gives rise to nontrivial wave-number selection even in the absence of the dispersion. As for the group velocity c, I will consider the general situation when it is not assumed small. It will be shown that the result for $c=0$ can be directly obtained from the general one by taking the limit $c \to 0$.

According to what was said above, all the unknown quantities $a(x)$, $k(x)$, and ω should be expanded in powers of α, β, and $\chi(x)$. At the zeroth order of the expansion, Eq. (4a) simplifies to

$$\epsilon(x)a_0 - ca_0' - a_0^3 + a_0'' = 0 \tag{5}$$

(hereafter, the subscripts refer to the order of the expansion). Equation (5) should be supplemented by the obvious boundary conditions (BC),

$$a_0(-\infty) = 0, \quad a_0(+\infty) = 1 \tag{6}$$

[recall $\epsilon(+\infty) = 1$]. At the same order, Eq. (4b) takes the form

$$\frac{d}{dx}(a_0^2 k_0) - c(a_0^2 k_0) = -\omega_0 a_0^2 \ . \tag{7}$$

Setting $x = +\infty$ in Eq. (7) yields the relation

$$\omega_0 = cK_0 \ , \tag{8}$$

where $K \equiv k(x = +\infty)$ is the asymptotic wave number sought for. Integration of Eq. (7) yields, with regard to Eq. (8),

$$a_0^2(x) k_0(x) = -cK_0 e^{cx} \int_{-\infty}^{x} e^{-cx'} a_0^2(x') dx' \ . \tag{9}$$

If c is negative, the integral in Eq. (9) diverges at large x, and in the limit $x \to +\infty$ Eq. (9) degenerates into the trivial identity $K = K$. This implies that there is no wave-number selection at negative c (i.e., if the group velocity points to the subcritical region). The situation is different if c is positive. In this case, the integral converges, and at large x Eq. (9) takes the asymptotic form

$$a_0^2(+\infty) K_0 = -cK_0 e^{cx} \int_{-\infty}^{+\infty} e^{cx'} a_0^2(x') dx' \ . \tag{10}$$

Since the right-hand side of Eq. (10) diverges at $x \to +\infty$, the only way to obtain a meaningful result is to set $K_0 = 0$. Thus, a positive finite c, as well as $c = 0$ [3], select $K_0 = 0$. Then it follows from Eqs. (7) and (8) that $k_0(x) \equiv 0$.

In the next approximation, Eq. (4b) takes the form

$$\frac{d}{dx}(a_0^2 k_1) - c(a_0^2 k_1) = -\omega_1 a_0^2 + \chi(x) a_0^2 + \alpha a_0^4 - \beta a_0 a_0'' \ . \tag{11}$$

At $x \to +\infty$, Eq. (11) yields [recall $\chi(+\infty) = 0$]

$$\omega_1 = cK_1 + \alpha \ . \tag{12}$$

According to what was said above, we consider only $c > 0$. In this case, straightforward integration of Eq. (11) yields, at $x \to +\infty$ [cf. Eq. (9)],

$$K_1 = e^{cx} \int_{-\infty}^{+\infty} e^{-cx'} [-(cK_1 + \alpha) a_0^2(x') + \chi(x') a_0^2(x') + \alpha a_0^4(x') - \beta a_0(x') a_0''(x')] dx' \ . \tag{13}$$

Since c is positive, the integral in Eq. (13) converges [11], while the preintegral factor, $\exp(cx)$, diverges at $x \to +\infty$. To compensate the divergence, one has to nullify the integral. This immediately yields

$$K_1 = -\alpha/c + (1/c) \left[\int_{-\infty}^{+\infty} e^{-cx} a_0^2(x) dx \right]^{-1} \int_{-\infty}^{+\infty} e^{-cx} [\chi(x) a_0^2(x) + \alpha a_0^4(x) - \beta a_0''(x) a_0(x)] dx \ . \tag{14}$$

Given a solution of the boundary problem (5) and (6), Eq. (14) solves the wave-number selection problem for the traveling waves.

Although only strictly positive c were considered, Eq. (14) admits a limit transition to the case $c = 0$. Indeed, a straightforward analysis demonstrates that, in this limit, the divergent multipliers c^{-1} are canceled by the divergence of the integrals involved, and the final result for $c = 0$ is

$$K_1 = \int_{-\infty}^{+\infty} \{\chi(x) a_0^2(x) - \alpha a_0^2(x)[1 - a_0^2(x)] + \beta [a_0'(x)]^2\} dx \ , \tag{15}$$

where the last term has been integrated by parts. On the other hand, Eq. (15) can be obtained by direct integration of Eq. (11) if one sets in it $c = 0$.

It is noteworthy that, if $\alpha = \beta = 0$ (no dispersion), but $\chi(x) \neq 0$, Eq. (14) still gives a nontrivial result,

$$K_1 = (1/c) \left[\int_{\infty}^{+\infty} e^{-cx} a_0^2(x) dx \right]^{-1} \times \int_{-\infty}^{+\infty} e^{-cx} \chi(x) a_0^2(x) dx \ . \tag{16}$$

This result is, actually, exact for the dispersionless system. Indeed, setting $\alpha = \beta = 0$, one notes that Eq. (4a) exactly coincides with Eq. (5), while Eq. (4b) coincides with Eq. (11). Thus, Eq. (16), obtained by integration of Eq. (11) with $\alpha = \beta = 0$, yields the exact expression for the selected wave number.

If, in addition to assuming α, β, and $\chi(x)$ small, one also assumes that the coefficients $\epsilon(x)$ and $\chi(x)$ vary at a large scale $l \gg 1$ [see, e.g., Eq. (2)], an approximate solution to Eq. (5) satisfying the BC (6) can be taken in the obvious form

$$a_0=0,\quad x<0;\quad a_0=\sqrt{\epsilon(x)},\quad x>0\ . \tag{17}$$

In this approximation, one may neglect the last term of the integrand in the numerator of Eq. (14), and then Eq. (14) takes the fully explicit form

$$K_1=-\alpha/c+(1/c)\left[\int_0^{+\infty}e^{-cx}\epsilon(x)dx\right]^{-1}\times\int_0^{+\infty}e^{-cx}[\chi(x)\epsilon(x)+\alpha\epsilon^2(x)]\ . \tag{18}$$

Equation (15), which pertains to the limiting case $c=0$, can also be simplified under the condition $l\gg 1$. Inserting Eq. (17) and neglecting the last term in Eq. (15), one readily finds

$$K_1=\int_0^{+\infty}\{\chi(x)\epsilon(x)-\alpha\epsilon(x)[1-\epsilon(x)]\}dx\ . \tag{19}$$

If, in particular, $\epsilon(x)$ is given by Eq. (2) and $\chi(x)=0$, $K_1=-(\pi-2)\alpha l$. Since the characteristic scale l of the function $\epsilon(x)$ is large, a similar estimate is true in the general case:

$$K\sim-\alpha l \tag{20}$$

[it is evident how to modify this estimate if the contribution from $\chi(x)$ is taken into account]. A significant feature of the expression (20) is that the smallness of α can be in part compensated by the large multiplier l. Actually, the same estimate is valid when c is finite but much smaller than $1/l$.

The analysis developed above assumed tacitly that the selected wave number complied with the stability conditions for traveling waves governed by the Ginzburg-Landau equation (1) with constant coefficients ($\epsilon\equiv 1$, $\chi\equiv 0$). The necessary stability conditions are [12,13] $1+\alpha\beta>0$, and

$$K^2<(1+\alpha\beta)(3+2\alpha^2+\alpha\beta)^{-1} \tag{21}$$

(these necessary conditions become sufficient if, in addition, $K^2<\frac{1}{3}$ [13]). It follows from Eq. (20) that if c is sufficiently small while l is sufficiently large, the wave number given by Eq. (19) may lie outside the stability band (21). Of course, the above analysis preassumed that K was small enough, but it seems quite plausible that the selected wave number may indeed get outside the stability region.

If this happens, it is natural to expect that the ramp generates a wave "turbulence" (dynamical chaos). Very preliminary results of numerical simulations of Eq. (1) demonstrate that the chaos does take place [10]. The onset of chaos induced by the ramp would not be very surprising because it is known that a spatial inhomogeneity in the Ginzburg-Landau equation is apt to generate a dynamical chaos [14,15].

Another nontrivial situation may take place (at least, in the particular case $c=0$) when a supercritical "island" is created in the subcritical bulk (as, e.g., in the annular system which is a good tool for the experiments with the traveling-wave convection [8]). In this case, a sufficiently long supercritical island is expected to generate waves traveling in the opposite directions. Thus, one should have a source of waves on the island. The corresponding quasistationary [i.e., having the general form of Eq. (3)] stable solution to Eq. (1) with the constant coefficients has been found in the geometric-optics (eikonal) approximation in Refs. [16] and [13]. If, however, the two ramps matching the island with the supercritical bulk select unequal wave numbers, the source must move with a velocity proportional to their difference. Since the island, although assumed long, has a finite size, the situation cannot be quasistationary even if the selected wave numbers belong to the stability band.

In conclusion, let us discuss the same problem for systems with the inverted bifurcation, when a traveling wave can be triggered by a finite disturbance in the subcritical region. As was mentioned above, this is a generic case for the traveling-wave convection [5–8]. The simplest model of the inverted bifurcation is based on the quintic Ginzburg-Landau equation. As was demonstrated in Ref. [17], this bistable model, unlike the monostable cubic one, admits a traveling-front solution. The front is a boundary between the trivial state and a traveling wave. The front's velocity is a uniquely determined function of all the parameters of the model, and there is a certain negative value ϵ_0 of the overcriticality at which the velocity vanishes; ϵ_0 is a function of the other parameters. In the system with the ramped overcriticality $\epsilon(x)$, the front should come to the point x_0 at which $\epsilon(x)=\epsilon_0$ and be stuck there, provided that the scale l at which $\epsilon(x)$ varies is much larger than a width of the front. It is natural to expect that the pinned state of the front at $x=x_0$ is stable. Next, it is known that the front uniquely selects the wave number and the frequency of the traveling wave [17]. The wave beyond the pinned front will then slowly evolve in x, adjusting itself to the slowly varying overcriticality. The asymptotic wave number at $x=+\infty$ will be different from that immediately selected by the front; however, it can be easily found since in a stationary state the frequency selected by the front should be spatially homogeneous. This scenario of the frequency and wave-number selection for the bistable systems seems much simpler than that in the monostable model based on Eq. (1).

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